Stat 534: formulae referenced in lecture, week 9, part 2: Population modeling

Song sparrows, 3 year model

- Eigenvalues and vectors are:

	Largest	2nd and 3rd largest, a pair	
eigenvalue	0.918	-0.459 + 0.335i	-0.459 - 0.335i
eigenvector	0.969	0.895 + 0.000i	0.895 + 0.000i
	0.211	-0.254 - 0.186i	-0.254 + 0.186i
	0.131	0.096 + 0.301i	0.096 - 0.301i

- 2nd and 3rd eigenvalues and vectors are complex conjugate pairs
- Ensures that complex parts cancel out when computing things like population counts
- Notice that largest eigenvalue is a real number - Perron-Froebenius Theorem

So why does only largest eigenvalue and associated matter for long term population growth?

- $\boldsymbol{N}_k = \boldsymbol{A}^k \boldsymbol{N}_0 = \boldsymbol{U}\boldsymbol{D}^k\boldsymbol{U}'\boldsymbol{N}_0$
- Divide and multiply by largest eigenvalue

$$\boldsymbol{N}_k = (\lambda^1)^k \boldsymbol{U} \begin{bmatrix} \left(\frac{\lambda^1}{\lambda^1}\right)^k & 0 & 0 \\ 0 & \left(\frac{\lambda^2}{\lambda^1}\right)^k & 0 \\ 0 & 0 & \left(\frac{\lambda^3}{\lambda^1}\right)^k \end{bmatrix} \boldsymbol{U}'\boldsymbol{N}_0$$

$$\boldsymbol{N}_k = (\lambda^1)^k \boldsymbol{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{\lambda^2}{\lambda^1}\right)^k & 0 \\ 0 & 0 & \left(\frac{\lambda^3}{\lambda^1}\right)^k \end{bmatrix} \boldsymbol{U}'\boldsymbol{N}_0$$

- Since $\lambda^2 < \lambda^1$ and $\lambda^3 < \lambda^1$, $\left(\frac{\lambda^2}{\lambda^1}\right)^k \to 0$, and $\left(\frac{\lambda^3}{\lambda^1}\right)^k \to 0$
- λ^1 controls the rate of population growth

- because contributions associated with λ_2 and λ_3 go to zero as $k \to$ large.

What's going on with those complex numbers?

- There is a close relationship between complex numbers and polar coordinates
- Which can be used to show that the initial dynamics are oscillations
  - If $\lambda_2 = a + b\,i$, period of oscillation is $2\pi/\tan^{-1}(b/a)$
- and the contribution from u^2 declines exponentially at rate $\lambda_1/\mid\lambda^2\mid$
  - where $\mid\lambda^2\mid$ is the modulus of $\lambda^2 = \sqrt{a^2+b^2}$
  - $\lambda_1/\mid\lambda^2\mid$ called the damping ratio $= d$

Song sparrows again:

	Largest	2nd largest pair	
eigenvalue	0.918	-0.459 + 0.335i	-0.459-0.335i
modulus	0.918	0.568	0.568

- Damping ratio = 0.918/0.568 = 1.62
- larger damping ratio $\Rightarrow$ oscillations die out more quickly
  - contribution of u^2 and u^3 to $\boldsymbol{N}_k$ after k years will be d^{-k}
  - after 1 years, contribution of u^2 is $1/d$ = 62% of what it was to $\boldsymbol{N}_0$
  - after 10 years, contribution of u^2 is $(1/d)^{10}$ = 0.8% of what it was to $\boldsymbol{N}_0$

Sensitivity analysis: Which matrix elements "matter"?

- In general, if we change the value of one matrix element by a little bit, how much does λ change?
  - Notation: $\lambda(\boldsymbol{A})$ dominant eigenvalue computed from matrix $\boldsymbol{A}$

- Define $\boldsymbol{P}$ as a matrix of 0's except for a single 1 for the element we want to change
- e.g.: Change f_1 a bit:

$$\boldsymbol{A} = \begin{bmatrix} 0 & f_1 & f_2 \\ \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What we want is

$$\frac{d\lambda(\boldsymbol{A})}{da_{ij}} = \frac{\lambda(\boldsymbol{A} + \delta\boldsymbol{P}) - \lambda(\boldsymbol{A})}{\delta}$$

  for small δ.
- More carefully, the limit of this as $\delta \to 0$

- Sensitivity: additive change in a matrix element, e.g. add 0.01

$$S_{ij} = \frac{v_i\, u_j}{\boldsymbol{v}'\boldsymbol{u}}$$

  - where $\boldsymbol{u}$ is the stable population distribution and $\boldsymbol{v}$ is the reproductive values.
  - Can be computed for any transition, even those that are biologically impossible
  - Answers a "what if it could be increased" question.

- Elasticity: proportional change in a matrix element, e.g. multiply by 1.1

$$E_{ij} = \frac{d \log \lambda(\boldsymbol{A})}{d \log a_{ij}} = S_{ij} \frac{a_{ij}}{\lambda(\boldsymbol{A})}$$

- Common to arrange sensitivities and elasticities into a matrix

  - Can compute all at once using an outer product

$$\boldsymbol{S} = \frac{\boldsymbol{v}\boldsymbol{u}'}{\boldsymbol{v}'\boldsymbol{u}}$$

  - and element by element = Hadamard multiplication

$$\boldsymbol{E} = \boldsymbol{S} \circ \boldsymbol{A}/\lambda(\boldsymbol{A})$$

Song sparrows again

- Sensitivity matrix:

$$\boldsymbol{S} = \begin{bmatrix} 0.42 & 0.09 & 0.06 \\ 1.93 & 0.42 & 0.26 \\ 1.19 & 0.26 & 0.16 \end{bmatrix}$$

- Elasticity matrix:

$$\boldsymbol{E} = \begin{bmatrix} 0.00 & 0.26 & 0.16 \\ 0.42 & 0.00 & 0.00 \\ 0.00 & 0.16 & 0.00 \end{bmatrix}$$